

COMMENTS ON THE MARTINGALE CONVERGENCE THEOREM  
(Preliminary Report)

by

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# Comments on the Martingale Convergence Theorem

S. D. Chatterji\*

Let  $(\Omega, \mathcal{B}, P)$  be a probability space and let  $X$  be a Banach space. A sequence of  $X$ -valued Bochner-integrable random variables  $f_n$  on  $\Omega$  will be said to form a martingale with respect to the subalgebras  $A_n, n=1, 2, \dots, A_n \subset A_{n+1}$  (in short  $\{f_n, A_n\}$  is a martingale) if

$$E^{A_n} f_{n+1} = f_n \quad n \geq 1,$$

where  $E^{A_n}$  is the conditional expectation operator with respect to the  $\sigma$ -algebra  $A_n$ . It is known that these operators are well-defined for arbitrary Banach-space-valued integrable functions. In the following it will be assumed that the algebra  $A = \bigvee_{n=1}^{\infty} A_n$  generates the  $\sigma$ -algebra  $\mathcal{B}$ . The general case can be handled via standard reduction to this case.

My main concern will be proving almost everywhere (a.e.) convergence theorems for martingales. For the sake of brevity, I shall limit myself in this talk to considering only the following statements:

- (S<sub>1</sub>) If  $f_n = E^{A_n} f$  then  $\lim_{n \rightarrow \infty} f_n = f$  a.e. (strong limit in  $X$ );
- (S<sub>2</sub>) If  $\{f_n, A_n\}$  is a martingale and the  $f_n$ 's are uniformly integrable (i.e.  $\lim_{n \rightarrow \infty} \int \|f_n\| \cdot I\{\|f_n\| > N\} dP = 0$  uniformly in  $n \geq 1$ ) then  $\exists f_{\infty}$  such that  $\lim_{n \rightarrow \infty} f_n = f_{\infty}$  a.e.;
- (S<sub>3</sub>) If  $\{f_n, A_n\}$  is a martingale with  $\sup_{n \geq 1} E(\|f_n\|) < \infty$  then  $\exists f_{\infty}$  such that  $\lim_{n \rightarrow \infty} f_n = f_{\infty}$  a.e.

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In section 2 I shall prove that  $(S_1)$  is always true. In this generality the result is proved by other methods in Chatterji (2b) and also in A.I. and C.I. Tulcea (6). The present proof, paralleling the proof in the scalar-valued case as in Billingsley (1), is as simple (possibly, some would wish to say, trivial) as one could wish for.

In section 3 I shall prove the main theorem of this paper, viz., that if  $X$  satisfies the following (RN) condition (RN for Radon-Nikodym) then  $(S_3)$  (and hence trivially  $(S_2)$ ) is valid for all martingales. The condition referred to is:

(RN): Every  $\sigma$ -additive  $X$ -valued set function  $\mu$  on  $\mathcal{B}$  of bounded variation with the property that  $V_\mu$ , the variation of  $\mu$ , is absolutely continuous with respect to  $P$  ( $V_\mu \ll P$ ) can be represented as the indefinite integral of an  $X$ -valued Bochner-integrable function.

The non-negative measure  $V_\mu$  is defined as follows:

$$V_\mu(A) = \sup \left\{ \sum_{i=1}^n \|\mu(A_i)\| : A_i A_j = \emptyset, A_i \in \mathcal{B}, \sum_{i=1}^n A_i = A, n \geq 1 \right\}.$$

The implication  $(RN) \Rightarrow (S_3)$  (more precisely  $X$  has (RN) property with respect to  $\mathcal{B}$  and  $P$  implies  $(S_3)$ ) is more general than the statements obtainable from (6). It also follows that  $(S_3)$  is valid for reflexive  $X$  or separable dual spaces  $X$ , statements explicitly made in (6). For reflexive  $X$ ,  $(S_2)$  (weaker than  $(S_3)$ ) was proved by different methods in (2a,b) and by Scalora (5). That some condition on  $X$  is necessary for the validity of  $(S_2)$  or  $(S_3)$  is demonstrated by the counterexample in (2a). Here a martingale  $f_n$  is constructed which takes values in  $L^1(0,1)$  and which does not converge in any sense, weak or strong, anywhere,

although among other things,  $\|f_n\| \equiv 1$  for all  $n \geq 1$ .

In section 4 it is shown that the (RN) condition is also necessary if  $B$  is separable (generated by a denumerable class of subsets). More precisely, in this case  $(S_2)$ ,  $(S_3)$  and (RN) are equivalent conditions.

Section 2: The main probabilistic tool is the following lemma:

Lemma 1: For any martingale  $\{f_n, A_n\}$ , if  $A \in A_{n_0}$  and  $\epsilon > 0$  then

$$P\{A: \sup_{k \geq n_0} \|f_k\| > \epsilon\} \leq \frac{1}{\epsilon} \left[ \sup_{k \geq n_0} \int_A \|f_k\| dP \right].$$

The lemma is known and an easy consequence of the fact that

$\|f_n\|$  is a submartingale.

Theorem 1  $(S_1)$  : For any space  $X$ ,  $\lim_{n \rightarrow \infty} E^{A_n} f = f$  a.e. (P).

Sketch of proof: If  $f$  is measurable  $A = \bigcup_{n=1}^{\infty} A_n$  then  $(S_1)$  is trivial since  $E^{A_n} f = f$  for sufficiently large  $n$ .

For a general  $f$ ,  $\exists f_\epsilon$  measurable in  $A$  such that

$$E(\|f - f_\epsilon\|) < \epsilon.$$

The following obvious inequality

$$\|E^{A_n} f - E^{A_m} f\| \leq \|E^{A_n} f_\epsilon - E^{A_m} f_\epsilon\| + 2 \sup_{k \geq 1} E^{A_k} \|f - f_\epsilon\|$$

coupled with lemma 1 leads us to  $(S_1)$  quite smoothly.

Section 3: Given a martingale  $\{f_n, A_n\}$ , define the set-functions on  $A_n$  as follows:

$$\mu_n(A) = \int_A f_n dP.$$

The martingale property is equivalent to the property that  $\mu_{n+1}$  is an extension of  $\mu_n$  to  $A_{n+1}$ . Hence for any  $A \in A = \bigcup_{n=1}^{\infty} A_n$   $\lim_{n \rightarrow \infty} \mu_n(A) = \mu(A)$  exists. The set-function  $\mu$  on  $A$  is an  $X$ -valued finitely additive set-function. The set-function is of bounded

variation iff  $\sup_{n \geq 1} \int \|f_n\| dP < \infty$ . The main difficulty in proving martingale convergence theorems is that  $\mu$  may not be  $\sigma$ -additive. The following lemma gives a way out.

Lemma 2: Let  $P$  be a probability measure on an algebra  $A$  of subsets of a space  $\Omega$  and  $\mu$  a finitely additive  $X$ -valued set function of bounded variation on  $A$ . Then

$$\mu = \eta + \sigma$$

where  $\eta, \sigma$  are both of bounded variation and  $\eta$  is a finitely additive set-function such that  $V_\eta$  (the variation of  $\eta$ ) is singular with respect to  $P$  (i.e. given  $\epsilon, \delta > 0, \exists A \in A, P(A) < \epsilon, V_\eta(A^c) < \delta$ ) and  $\sigma$  is a  $\sigma$ -additive set-function such that  $V_\sigma$  is absolutely continuous with respect to  $P$  (i.e. given  $\epsilon > 0, \exists \delta > 0, P(A) < \delta \Rightarrow V_\sigma(A) < \epsilon$ ).

The main idea behind the proof of the lemma will be sketched. One transfers  $P$  and  $\mu$  to the space  $(S, \mathcal{L}_1)$  where  $S$  is a totally disconnected compact Hausdorff space and  $\mathcal{L}_1$  is the algebra of clopen sets in  $S$ ,  $\mathcal{L}_1$  being isomorphic to  $A$ . It turns out that  $P$  and  $\mu$  are  $\sigma$ -additive on  $\mathcal{L}_1$  and hence can be extended to the  $\sigma$ -algebra  $\mathcal{L}_2$  generated by  $\mathcal{L}_1$ . (These are standard methods in this sort of work. See e.g. (3), pp. 312-13). On these extended measures on  $\mathcal{L}_2$  one applies the Lebesgue decomposition theorem as proved by Rickart (4) and one retraces the way back through  $\mathcal{L}_1$  to  $A$  to obtain the decomposition indicated in the lemma.

With the help of lemma 2, I shall now prove the main theorem of this talk.

Theorem 2: If  $X$  satisfies the (RN) property with respect to  $P$  on  $\mathcal{B}$  then any martingale  $\{f_n, \mathcal{A}_n\}$  with  $\sup_{n \geq 1} \int \|f_n\| dP$  converges;

i.e.  $\exists f_\infty$  such that  $\lim_{n \rightarrow \infty} f_n = f_\infty$  a.e. (P).

Sketch of the Proof: Let  $\mu$  be as before and  $\eta, \sigma$  as in lemma 2,  $\mu$  restricted to  $A_n$  is an integral.  $\sigma$ , being absolutely continuous with respect to P, is also an integral since X has the (RN) property.

Let  $\sigma(A) = \int_A h \, dP, \quad A \in \mathcal{A},$

and  $\sigma(A) = \sigma_n(A) = \int_A h_n \, dP, \quad A \in \mathcal{A}_n$

Clearly  $h_n = E^{A_n} h$ .

Hence  $\eta$  restricted to  $\mathcal{A}_n$  is also an integral, i.e.

$$\eta(A) = \eta_n(A) = \int_A g_n \, dP, \quad A \in \mathcal{A}_n$$

In other words,  $f_n = g_n + h_n$ ,

where  $g_n, h_n$  are also martingales with respect to  $\mathcal{A}_n$ .

Moreover  $h_n$  is  $E^{A_n} h$  and hence theorem 1 ensures the convergence of  $h_n$  to a limit.

I shall now show that  $\lim g_n = 0$  a.e. (P).

Given  $1 > \epsilon, \delta > 0$ , find  $A \in \mathcal{A}$  (and hence  $A \in \mathcal{A}_{n_0}$  for some  $n_0$ ) such that

$$P(A') + V_\eta(A) < \frac{\delta \epsilon}{2}.$$

Now

$$\begin{aligned} P\left\{\sup_{n \geq n_0} \|g_n\| > \epsilon\right\} &= P\{A': \sup_{n \geq n_0} \|g_n\| > \epsilon\} + P\{A: \sup_{n \geq n_0} \|g_n\| > \epsilon\} \\ &< \frac{\delta \epsilon}{2} + \frac{1}{\epsilon} \sup_{n \geq n_0} \int_A \|g_n\| \, dP \quad (\text{by lemma 1}) \\ &< \frac{\delta \epsilon}{2} + \frac{1}{\epsilon} V_\eta(A) < \frac{\delta \epsilon}{2} + \frac{\delta}{2} < \delta. \end{aligned}$$

This is clearly enough to show that  $\lim g_n = 0$  a.e. (P).

Section 4: In this section, the main thing is the following lemma for real-valued submartingales:

Lemma 3: If  $\{g_n, \mathcal{A}_n\}$  is a positive submartingale with  $\sup_{n \geq 1} E(g_n) < \infty$

such that  $\mu(A) = \lim_{n \rightarrow \infty} \int_A g_n dP$ ,  $A \in \mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{A}_n$ ,

is a  $\sigma$ -additive  $P$ -continuous set-function; then the  $g_n$ 's are uniformly integrable.

Sketch of Proof: If  $g_n \geq 0$  is a martingale then it is easy.

In general  $\exists h_n$  a martingale such that  $0 \leq g_n \leq h_n$  and such that  $\{h_n\}$  induces the same  $\mu$ . Hence the lemma.

Theorem 3: If  $\mathcal{B}$  is separable then  $(S_2) \Rightarrow (RN)$ . Hence in this case  $(S_2) \Leftrightarrow (S_3) \Leftrightarrow (RN)$ .

Sketch of Proof: Let  $\mathcal{B}$  be generated by  $A_1, A_2, \dots$  and  $\mathcal{A}_n$  equal to the  $\sigma$ -algebra generated by  $A_1, A_2, \dots, A_n$ . Given a set-function  $\mu$  on  $\mathcal{B}$  satisfying the conditions in  $(RN)$ , the martingale  $\{f_n, \mathcal{A}_n\}$  induced by  $\mu$  is such that  $\|f_n\|$  satisfies the conditions of lemma 3. Hence  $(S_2)$  implies the convergence of  $f_n$  to  $f_\infty$ . From here on it is trivial to show that  $\mu$  is the indefinite integral of  $f_\infty$ .

Note: In the real-valued case the general martingale convergence theorem  $S_3$  can be deduced rapidly from  $S_1$  by the following sequence of arguments:

(I)  $(S_1) \Rightarrow (S_2)$  because  $f_n$  uniformly integrable implies that  $\exists n_k$  such that  $f_{n_k} \rightarrow f$  weakly in  $L^1$  for some  $f$ . Hence  $E^{A_{n_k}} f_{n_k} \xrightarrow{k \rightarrow \infty} E^{A_{n_k}} f$  weakly. But  $E^{A_{n_k}} f_{n_k} = f_{n_k}$  for large  $n_k$ . Hence  $E^{A_{n_k}} f = f_{n_k}$ , etc.

(II) every uniformly integrable submartingale converges: this follows from (I) via the Doob-decomposition for submartingales.

(III) Every positive martingale  $f_n$  converges since  $e^{-f_n}$  is a uniformly bounded semimartingale.

(IV) An arbitrary martingale  $f_n$  with  $\sup E f_n^+ < \infty$  converges because it is the difference of two positive martingales and (III).

From here the same theorem for submartingales can also be easily obtained.



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